Homologcal algebra exercise sheet Week 3

Throughout the whole sheet, $\mathcal{A}$ will denote an abelian category.
Remember that a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in an abelian category is said to be *exact* at B if $g \circ f = 0$ and the induced map $\operatorname{im}(f) \to \ker(g)$ is an isomorphism.

1. Let $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ be a sequence of chain complexes in $\mathcal{A}$. Show, without using Freyd-Mitchell embedding theorem, that if

$$0 \to A_n \to B_n \to C_n \to 0$$

is exact in $\mathcal{A}$ for all $n \in \mathbb{Z}$, then

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is exact in $\mathbf{Ch}(\mathcal{A})$.

2. Let $f_\bullet : B_\bullet \to C_\bullet$ and $g_\bullet : D_\bullet \to C_\bullet$ be morphisms of chain complexes. Assume that $f_\bullet$ is monic and that, for each $n \in \mathbb{Z}$, there exists a map $h_n : D_n \to B_n$ such that $f_n \circ h_n = g_n$. Show that it defines a morphism $h_\bullet$, such that $f_\bullet \circ h_\bullet = g_\bullet$ in the category of chain complexes. In other words, show that if the following diagram commutes in $\mathcal{A}$ for every $n \in \mathbb{Z}$,

$$\begin{tikzcd} B_n \arrow[rr, "f_n"] & & C_n \\ & D_n \arrow[ul, dashed, "\exists h_n"] \arrow[ur, "g_n"'] & \end{tikzcd}$$

then $h_\bullet$ is a morphism and the following commutes in $\mathbf{Ch}(\mathcal{A})$:

$$\begin{tikzcd} B_\bullet \arrow[rr, "f_\bullet"] & & C_\bullet \\ & D_\bullet \arrow[ul, dashed, "h_\bullet"] \arrow[ur, "g_\bullet"'] & \end{tikzcd}$$

3. Let $D_{\bullet,\bullet} = \{D_{p,q}\}$ be a bounded double complex with maps

$$d^h_{p,q} = d^h : C_{p,q} \to C_{p-1,q} \quad \text{and} \quad d^v_{p,q} = d^v : C_{p,q} \to C_{p,q-1}.$$

Show that, if each row (or each column) $D_{\bullet,q}$ is an exact sequence, then the total complex $\operatorname{Tot}(D_{\bullet,\bullet})$ is also exact.
Hint: You may need Freyd-Mitchell Embedding Theorem.

4. (Mapping Cone)

   (a) Let $f_\bullet : B_\bullet \to C_\bullet$ be a morphism of chain complexes. Show that $f_\bullet$ induces a natural double complex $D_{\bullet,\bullet}$ which contains exactly two non-trivial rows: $B_\bullet$ and $C_\bullet$.

   (b) We have seen that for every double complex $D_{\bullet,\bullet}$, by adding signs to all vertical maps $d^h_{p,q}$, one gets the maps $f_{p,q} := (-1)^p d^v_{p,q}$. This defines a morphism of chain complexes : $f_{\bullet,q} : C_{\bullet,q} \to C_{\bullet,q-1}$. Therefore, we may view $\{C_{\bullet,q}\}_{q\in\mathbb{Z}}$ as a chain complex over the category $\mathbf{Ch}(\mathcal{A})$ and $f_{\bullet,q}$ as the boundary map at the q-th position. Show that this sign trick gives us a 1-1 correspondence between the objects of $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$ and the collection of all double complexes over $\mathcal{A}$.

   (c) Because of (b), we would like to define the category of double complexes in such that it is isomorphic to $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$. Thus, we define a *morphism of double complexes* between $C_{\bullet,\bullet}$ and $D_{\bullet,\bullet}$ as a collection of maps $\{f_{p,q} : C_{p,q} \to D_{p,q}\}$, such that after turning $C_{\bullet,\bullet}, D_{\bullet,\bullet}$ into objects in $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$ and using the sign trick, $f_{\bullet,\bullet}$ is a morphism of $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$.

   Now let $B[-1]_\bullet$ be the chain complex defined by $B[-1]_n = B_{n-1}$ with differentials $-d$, where d are the differentials of B (see **Translation 1.2.8** in Weibel's book for a more general definition).

   Show that one can consider $C_\bullet$ and $B[-1]_\bullet$ as double complexes in a natural way. Moreover, prove that we have the following exact sequence in the category of double complexes:

$$0 \to C_\bullet \to D_{\bullet,\bullet} \to B[-1]_\bullet \to 0.$$

Remark: The total complex of $D_{\bullet,\bullet}$ is the *mapping cone* of f', denoted by $\mathrm{cone}(f')$, where f' is a map which differs from f by some signs. We will encounter the mapping cone later (see section 1.5 in Weibel's book).